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Universal Overconvergence of Polynomial Expansions of Harmonic Functions

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For each compact subset K of $\mathbb{R}^N$ let $\mathscr{H}(K)$ denote the space of functions that are harmonic on some neighbourhood of K. The space $\mathscr{H}(K)$ is equipped with the topology of uniform convergence on K. Let Ω be an open subset of $\mathbb{R}^N$ such that $0 \in \Omega$ and $\mathbb{R}^N \backslash \bar{\Omega}$ is connected. It is shown that there exists a series $\sum H_n$, where H_n is a homogeneous harmonic polynomial of degree n on $\mathbb{R}^N$, such that (i) $\sum H_n$ converges on some ball of centre 0 to a function that is continuous on $\bar{\Omega}$ and harmonic on Ω, (ii) the partial sums of $\sum H_n$ are dense in $\mathscr{H}(K)$ for every compact subset K of $\mathbb{R}^N \backslash \bar{\Omega}$ with connected complement. Some refinements are given and our results are compared with an analogous theorem concerning overconvergence of power series.

Key Words: harmonic; polynomial; overconvergence; series; density; universal.

1. INTRODUCTION

The following theorem was proved in the case $r = 0$ by Seleznev [19] and in the general case by Luh [9] and Chui and Parnes [5]. For extensions and generalizations, see also [10–13, 15–17, 20–22] and for a survey of related work, see [8].

THEOREM A. *Let $0 \leqslant r < +\infty$. There exists a power series $\sum_{n=0}^{\infty} a_n z^n$ of radius of convergence r such that for every compact set K in $\{z : |z| > r\}$ with connected complement and every function f that is continuous on K and holomorphic on the interior of K there exists an increasing sequence (n_k) of integers such that*

$$\sum_{n=0}^{n_k} a_n z^n \to f(z) \qquad (k \to \infty)$$

uniformly on K.

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The phenomenon exhibited by the power series in Theorem A is called overconvergence. Nestoridis [16,17] considers overconvergence of Taylor series of functions that are holomorphic on an arbitrary simply connected domain in $\mathbb{C}$ and which may be smooth up to the boundary. The purpose of this paper is to prove results of the same character as those in [16, 17] for harmonic functions on $\mathbb{R}^N$, where $N \geqslant 2$. First, we establish some notation. If E is a non-empty subset of $\mathbb{R}^N$, we denote by $\mathscr{H}(E)$ the space of functions h that are harmonic on some open set (which depends on h) containing E. In particular, in the case where E is open $h \in \mathscr{H}(E)$ if and only if h is harmonic on E. As usual, $C(E)$ denotes the space of real-valued continuous functions on E. For each $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, let $\mathscr{H}_n$ denote the space of all homogeneous harmonic polynomials of degree n on $\mathbb{R}^N$. If $h \in \mathscr{H}(B(r))$, where $B(r)$ is the open ball of radius r centred at the origin 0 of $\mathbb{R}^N$, then there exist unique polynomials $H_n \in \mathscr{H}_n$ such that $\sum_{n=0}^{\infty} H_n$ converges locally uniformly to h on $B(r)$ (see e.g. [2, p. 42]). We refer to $\sum_{n=0}^{\infty} H_n$ as the *polynomial expansion* of h, and if $\sum_{n=0}^{\infty} H_n$ is the polynomial expansion of some function $h \in C(\bar{\Omega}) \cap \mathscr{H}(\Omega)$, where Ω is a domain containing 0, then we write $\sum_{n=0}^{\infty} H_n \in \mathscr{PE}(\bar{\Omega})$.

THEOREM 1. *Let Ω be a bounded domain in $\mathbb{R}^N$ such that $0 \in \Omega$ and $\mathbb{R}^N \backslash \bar{\Omega}$ is connected. There exist harmonic polynomials $H_n \in \mathscr{H}_n$ such that $\sum_{n=0}^{\infty} H_n \in \mathscr{PE}(\bar{\Omega})$ and for every compact subset K of $\mathbb{R}^N \backslash \bar{\Omega}$ with connected complement and every $h \in \mathscr{H}(K)$ there exists an increasing sequence (n_k) in $\mathbb{N}$ such that*

$$\sum_{n=0}^{n_k} H_n \to h \qquad (k \to \infty) \tag{1}$$

uniformly on K.

The key condition concerning K and $\bar{\Omega}$ for the proof of Theorem 1 is that K lies in the unbounded component of $\mathbb{R}^N \backslash \bar{\Omega}$; it is not necessary to have $\mathbb{R}^N \backslash \bar{\Omega}$ connected. Thus an apparently, though not actually, more general formulation of the theorem is possible.

We write E° for the interior of a subset E of $\mathbb{C}$ or $\mathbb{R}^N$, and if $E \subseteq \mathbb{C}$, then $C(E)$ denotes the set of complex-valued continuous functions on E. The space of functions that are holomorphic on an open subset ω of $\mathbb{C}$ is denoted by $\mathrm{Hol}(\omega)$. For a compact subset K of $\mathbb{C}$, Mergelyan's theorem [14] (or see [18, Chap. 20]) asserts the equivalence of the following statements: (i) every function in $C(K) \cap \mathrm{Hol}(K^\circ)$ can be uniformly approximated by (holomorphic) polynomials, (ii) $\mathbb{C} \backslash K$ is connected. Thus, the conditions imposed on K and f in Theorem A are natural. In Theorem 1 the situation is quite different. The condition that $\mathbb{R}^N \backslash K$ is connected is neither necessary nor (when $N \geqslant 3$) sufficient for functions in $C(K) \cap \mathscr{H}(K^\circ)$ to be uniformly approximable on K by harmonic polynomials. The same condition is, however, sufficient (but not necessary) for functions in $\mathscr{H}(K)$ to be approximable in this way (cf. Lemmas 2 and 3 for necessary and sufficient conditions). In Section 3, we enlarge upon these remarks and examine the extent to which the hypotheses in Theorem 1 can be relaxed.

2. PROOF OF THEOREM 1

LEMMA 1. *Let Ω be a bounded domain in $\mathbb{R}^N$ such that $B(r) \subseteq \Omega$ for some $r > 0$ and $\mathbb{R}^N \backslash \bar{\Omega}$ is connected, and let K be a compact subset of $\mathbb{R}^N \backslash \bar{\Omega}$ such that $\mathbb{R}^N \backslash K$ is connected. If $\varepsilon > 0, R > 0, \mu \in \mathbb{N}$ and $G \in \mathscr{H}(\mathbb{R}^N)$, then there exists a harmonic polynomial $F = \sum_{j=0}^{m} f_j$, where $f_j \in \mathscr{H}_j$ and $m > \mu$, such that*

$$|F - G| < \varepsilon \qquad \text{on } K, \tag{2}$$

$$|F| < \varepsilon \qquad \text{on } \bar{\Omega}, \tag{3}$$

$$\sum_{j=0}^{\mu} |f_j| < \varepsilon \qquad \text{on } \overline{B(R)} \tag{4}$$

and

$$|f_j(x)| \leqslant \varepsilon (j+1)^{(N-2)/2} (\|x\|/r)^j \qquad (x \in \mathbb{R}^N, j = 0, 1, \ldots, m). \tag{5}$$

The proof of the lemma uses the fact that there is a constant $C_N \geqslant 1$, depending only on N, with the following property: if $u_j \in \mathscr{H}_j$ for $j = 0, 1, \ldots, v$, where $v \in \mathbb{N}$, and $r > 0$, then

$$\sup_{B(r)} |u_j| \leqslant C_N (j+1)^{(N-2)/2} \sup_{B(r)} |u| \qquad (j = 0, 1, \ldots, v), \tag{6}$$

where $u = \sum_{j=0}^{v} u_j$. In the case where $r = 1$ this follows easily from an inequality of Brelot and Choquet [4]; details are given in [1]. The general case follows from a simple dilation argument.

To proceed with the proof, let η be a positive number such that

$$C_N \eta \sum_{j=0}^{\mu} (j+1)^{(N-2)/2} (R/r)^j < \varepsilon, \tag{7}$$

which clearly implies that $C_N \eta < \varepsilon$. Let V_1, V_2 be disjoint open sets containing $\bar{\Omega}$ and K, respectively, and define a function G_0 by putting $G_0 = 0$ on V_1 and $G_0 = G$ on V_2. Then $G_0 \in \mathscr{H}(\bar{\Omega} \cup K)$. Since $\bar{\Omega} \cup K$ is compact and has connected complement, it follows from Walsh's harmonic approximation theorem [23] (or see [7, p. 8]) that there exists a harmonic polynomial F such that $|F - G_0| < \eta$ on $\bar{\Omega} \cup K$. Thus (2) and (3) hold, since

$\eta < \varepsilon$. Let $F = \sum_{j=0}^{m} f_j$, where $f_j \in \mathscr{H}_j$. We can arrange that $m > \mu$ simply by defining some of the f_j to be 0, if necessary. Since $|F| < \eta$ on $B(r)$, it follows from (6) that

$$|f_j(x)| < C_N \eta (j+1)^{(N-2)/2} \qquad (x \in B(r),\ j = 0, 1, \ldots, m),$$

and hence, by the homogeneity of f_j,

$$|f_j(x)| \leqslant C_N \eta (j+1)^{(N-2)/2} (\|x\|/r)^j \qquad (x \in \mathbb{R}^N,\ j = 0, 1, \ldots, m). \quad (8)$$

Since $C_N \eta < \varepsilon$, (5) follows. Also, summing (8) over $j = 0, 1, \ldots, \mu$ and using (7), we obtain (4).

We can now complete the proof of Theorem 1. Let $\mathscr{K}$ denote the set of compact sets K with the following properties: $K \subset \mathbb{R}^N \backslash \bar{\Omega}$; $\mathbb{R}^N \backslash K$ is connected; K is a finite union of closed cubes each with all its vertices at points with rational coordinates. The set $\mathscr{K}$ is countable; let (K_ν) be an enumeration of the elements of $\mathscr{K}$. Also, let (P_ν) be a sequence of harmonic polynomials that is dense in $\mathscr{H}(\mathbb{R}^N)$ with the topology of local uniform convergence. (For example, if $\mathscr{B}_j$ is a basis for $\mathscr{H}_j$, we could take (P_ν) to be an enumeration of all finite linear combinations with rational coefficients of elements of $\bigcup_{j=0}^{\infty} \mathscr{B}_j$.) Let $((Q_n, L_n))_{n \in \mathbb{N}}$ be a sequence of ordered pairs in which each pair (P_ν, K_μ) occurs. We show recursively that for each $\lambda \in \mathbb{N}$ there exist $m_\lambda \in \mathbb{N}$ and $F_\lambda = f_{\lambda,0} + \cdots + f_{\lambda, m_\lambda}$, where $f_{\lambda,n} \in \mathscr{H}_n$, such that (m_λ) is increasing and when $\lambda \geqslant 1$

$$|F_0 + \cdots + F_\lambda - Q_\lambda| < 2^{-\lambda} \qquad \text{on } L_\lambda, \quad (9)$$

$$|F_\lambda| < 2^{-\lambda} \qquad \text{on } \bar{\Omega}, \quad (10)$$

$$\sum_{n=0}^{m_{\lambda-1}} |f_{\lambda,n}| < 2^{-\lambda} \qquad \text{on } B(\lambda) \cup L_0 \cup \cdots \cup L_{\lambda-1} \quad (11)$$

and

$$|f_{\lambda,n}(x)| \leqslant 2^{-\lambda} (n+1)^{(N-2)/2} (\|x\|/r)^n \qquad (x \in \mathbb{R}^N,\ n = 0, 1, \ldots, m_\lambda), \quad (12)$$

where $r > 0$ is such that $B(r) \subseteq \Omega$. We start by defining $m_0 = 0$ and $F_0 = f_{0,0} = 0$. Suppose now that for some $\lambda \in \mathbb{N}$ the integers $m_0, m_1, \ldots, m_\lambda$ and harmonic polynomials $F_j = f_{j,0} + \cdots + f_{j,m_j}$ $(j = 0, 1, \ldots, \lambda)$ have been shown to exist. In Lemma 1, take $K = L_{\lambda+1}$ and $G = Q_{\lambda+1} - (F_0 + \cdots + F_\lambda)$, and let R be such that $B(\lambda+1) \cup L_0 \cup \cdots \cup L_\lambda \subset B(R)$. We find that there exists a harmonic polynomial $F_{\lambda+1} = f_{\lambda+1,0} + \cdots + f_{\lambda+1, m_{\lambda+1}}$, where $f_{\lambda+1,n} \in \mathscr{H}_n$ and $m_{\lambda+1} > m_\lambda$, such that (9)–(12) hold with $\lambda + 1$ in place of λ throughout. This completes the inductive step.

We define $f_{\lambda,n} = 0$ when $n > m_\lambda$ and write

$$H_n = \sum_{\lambda=0}^{\infty} f_{\lambda,n} \tag{13}$$

for each $n \in \mathbb{N}$. We claim that $H_n \in \mathscr{H}_n$. To verify this, let ρ be a positive number and note that if $\lambda > \rho + n + 1$, then $n < \lambda - 1 \leqslant m_{\lambda-1}$, so that $|f_{\lambda,n}| < 2^{-\lambda}$ on $B(\rho)$ by (11). Hence the series in (13) converges uniformly on $B(\rho)$ and, since ρ is arbitrary, it follows that $H_n \in \mathscr{H}(\mathbb{R}^N)$. Since each function $f_{\lambda,n}$ is homogeneous of degree n, so also is H_n, and hence $H_n \in \mathscr{H}_n$.

Next we prove that $\sum_{n=0}^{\infty} H_n \in \mathscr{PE}(\bar{\Omega})$. Inequality (10) shows that $\sum_{\lambda=0}^{\infty} F_\lambda$ converges uniformly on $\bar{\Omega}$. Hence the sum of this series, g say, belongs to $C(\bar{\Omega}) \cap \mathscr{H}(\Omega)$. If $x \in B(r)$, then by (12)

$$\sum_{n=0}^{\infty} \sum_{\lambda=0}^{\infty} |f_{\lambda,n}(x)| \leqslant \sum_{n=0}^{\infty} (n+1)^{(N-2)/2}(||x||/r)^n \sum_{\lambda=0}^{\infty} 2^{-\lambda} < +\infty.$$

Hence the following change of order is justified:

$$\sum_{n=0}^{\infty} H_n(x) = \sum_{n=0}^{\infty} \sum_{\lambda=0}^{\infty} f_{\lambda,n}(x) = \sum_{\lambda=0}^{\infty} \sum_{n=0}^{\infty} f_{\lambda,n}(x) = \sum_{\lambda=0}^{\infty} F_\lambda(x) = g(x) \qquad (x \in B(r)).$$

Since $H_n \in \mathscr{H}_n$ and the polynomial expansion of g is unique, it follows that $\sum_{n=0}^{\infty} H_n$ is the polynomial expansion of g, and hence $\sum_{n=0}^{\infty} H_n \in \mathscr{PE}(\bar{\Omega})$.

In preparation for the final part of the proof, we note that if $\lambda \geqslant 1$, then on L_λ

$$\begin{aligned}
\left| \sum_{n=0}^{m_\lambda} H_n - Q_\lambda \right| &= \left| \sum_{n=0}^{m_\lambda} \sum_{v=0}^{\infty} f_{v,n} - Q_\lambda \right| \\
&= \left| \sum_{v=0}^{\infty} \sum_{n=0}^{m_\lambda} f_{v,n} - Q_\lambda \right| \\
&\leqslant \left| \sum_{v=0}^{\lambda} \sum_{n=0}^{m_\lambda} f_{v,n} - Q_\lambda \right| + \left| \sum_{v=\lambda+1}^{\infty} \sum_{n=0}^{m_\lambda} f_{v,n} \right| \\
&\leqslant \left| \sum_{v=0}^{\lambda} F_v - Q_\lambda \right| + \sum_{v=\lambda+1}^{\infty} \sum_{n=0}^{m_v - 1} |f_{v,n}| \\
&< 2^{-\lambda} + \sum_{v=\lambda+1}^{\infty} 2^{-v} = 2^{1-\lambda},
\end{aligned}$$

by (9) and (11).

Now let K be a compact subset of $\mathbb{R}^N \backslash \bar{\Omega}$ with connected complement. We show that $K \subset K_\mu$ for some μ. Let Ω_0 be an unbounded domain such that $\Omega \subset \Omega_0$ and $K \subset \mathbb{R}^N \backslash \bar{\Omega}_0$, and let δ be a rational number such that $0 < \delta\sqrt{N} < \mathrm{dist}(K, \bar{\Omega}_0)$. Let $\mathscr{C}$ be the collection of closed cubes of side-length δ with vertices lying in the lattice $\{\delta y : y \in \mathbb{Z}^N\}$. Let M be the union of those elements of $\mathscr{C}$ that have non-empty intersection with K. Then $M \cap \bar{\Omega}_0 = \emptyset$, so $\bar{\Omega}$ lies in the unbounded component of $\mathbb{R}^N \backslash M$. Let $\hat{M}$ denote the union of M with the bounded components of $\mathbb{R}^N \backslash M$. Then $\hat{M}$ is a finite union of elements of $\mathscr{C}$ and $\mathbb{R}^N \backslash \hat{M}$ is connected and contains $\bar{\Omega}$. Thus $\hat{M} = K_\mu$ for some μ. Also $K \subseteq M \subseteq \hat{M}$.

Let $h \in \mathscr{H}(K)$ and let $\varepsilon > 0$. It is enough to show that there exists an arbitrarily large λ such that

$$\left| \sum_{n=0}^{m_\lambda} H_n - h \right| < \varepsilon \qquad \text{on} \quad K.$$

By Walsh's theorem, there exists $h_0 \in \mathscr{H}(\mathbb{R}^N)$ such that $|h - h_0| < \varepsilon/3$ on K. Since (P_ν) was chosen to be dense in $\mathscr{H}(\mathbb{R}^N)$, there exist infinitely many ν such that $|P_\nu - h_0| < \varepsilon/3$ on K. Hence there exists λ, as large as we please, such that $K \subseteq L_\lambda$, $|Q_\lambda - h_0| < \varepsilon/3$ on K, and $2^{1-\lambda} < \varepsilon/3$. Collecting results together, we find that on K

$$\left| \sum_{n=0}^{m_\lambda} H_n - h \right| \leqslant \left| \sum_{n=0}^{m_\lambda} H_n - Q_\lambda \right| + |Q_\lambda - h_0| + |h_0 - h|$$

$$= 2^{1-\lambda} + \varepsilon/3 + \varepsilon/3 < \varepsilon,$$

as required.

3. IMPROVEMENTS OF THEOREM 1

At the end of Section 1 we indicated that some relaxation is possible in the hypotheses concerning K and h in Theorem 1. By invoking some recent results on uniform harmonic approximation, which are stated below as lemmas, we now show precisely the ways in which these hypotheses can be relaxed. For the concept of thinness, which appears in the lemmas, we refer to [2, Chap. 7]. If K is a compact subset of $\mathbb{R}^N$, then we denote by $\hat{K}$ the union of K with all the bounded connected components of $\mathbb{R}^N \backslash K$.

LEMMA 2. *Let K be a compact subset of $\mathbb{R}^N$. The following are equivalent*:

(a) *for each u in $\mathscr{H}(K)$ and each positive number ε there exists v in $\mathscr{H}(\mathbb{R}^N)$ such that $|u - v| < \varepsilon$ on K*;

(b) *$\mathbb{R}^N \backslash \hat{K}$ and $\mathbb{R}^N \backslash K$ are thin at the same points of K.*

LEMMA 3. *Let K be a compact subset of $\mathbb{R}^N$. The following are equivalent*:

(a′) *for each u in $C(K) \cap \mathcal{H}(K^\circ)$ and each positive number ε there exists v in $\mathcal{H}(\mathbb{R}^N)$ such that $|u - v| < \varepsilon$ on K*;

(b′) *$\mathbb{R}^N \backslash \hat{K}$ and $\mathbb{R}^N \backslash K^\circ$ are thin at the same points of K.*

Lemma 2 is a refinement of Walsh's theorem and is the harmonic analogue of Runge's classical theorem on holomorphic approximation. Lemma 3 is the harmonic analogue of Mergelyan's theorem. Both lemmas are special cases of theorems of Gardiner [6], and they can also be derived from results of Bliedtner and Hansen [3]. A convenient reference is [7, Theorems 1.10 and 1.15]. Lemmas 2 and 3 easily yield the following improvements of Theorem 1.

COROLLARY 1. *Let Ω be a bounded domain in $\mathbb{R}^N$ such that $0 \in \Omega$ and $\mathbb{R}^N \backslash \bar{\Omega}$ is connected. There exist harmonic polynomials $H_n \in \mathcal{H}_n$ with the following properties*:

(i) $\sum_{n=0}^{\infty} H_n \in \mathcal{PE}(\bar{\Omega})$;

(ii) *for any compact set K such that $\hat{K} \subset \mathbb{R}^N \backslash \bar{\Omega}$ and $\mathbb{R}^N \backslash K$ and $\mathbb{R}^N \backslash \hat{K}$ are thin at the same points of K, and for any $h \in \mathcal{H}(K)$, there exists an increasing sequence (n_k) such that* (1) *holds uniformly on K.*

(iii) *for any compact set K such that $\hat{K} \subset \mathbb{R}^N \backslash \bar{\Omega}$ and $\mathbb{R}^N \backslash \hat{K}$ and $\mathbb{R}^N \backslash K^\circ$ are thin at the same points of K, and for any $h \in C(K) \cap \mathcal{H}(K^\circ)$, there exists an increasing sequence (n_k) such that* (1) *holds uniformly on K.*

To prove Corollary 1, let the harmonic polynomials H_n be as in Theorem 1 and suppose first that K and h are as stated in Corollary 1(ii). If $\varepsilon > 0$, then by Lemma 2, there exists $g \in \mathcal{H}(\mathbb{R}^N)$ such that $|g - h| < \varepsilon/2$ on K. Since $\hat{K}$ is a compact subset of $\mathbb{R}^N \backslash \bar{\Omega}$ and $\mathbb{R}^N \backslash \hat{K}$ is connected, there exist arbitrarily large integers μ such that $|\sum_{n=0}^{\mu} H_n - g| < \varepsilon/2$ on $\hat{K}$, and hence $|\sum_{n=0}^{\mu} H_n - h| < \varepsilon$ on K, as required. The same argument, with Lemma 3 replacing Lemma 2, shows that if K and h are as in Corollary 1(iii), then the same conclusion holds.

It is natural to ask whether the hypotheses on K and h in Theorem 1 can be made exactly analogous to the hypotheses on K and f in Theorem A: in Theorem 1 can we replace "$h \in \mathcal{H}(K)$" by "$h \in C(K) \cap \mathcal{H}(K^\circ)$"? When $N = 2$ the answer is affirmative, and we can even relax the condition that $\mathbb{R}^2 \backslash K$ is connected.

COROLLARY 2. *Let Ω be a bounded domain in $\mathbb{R}^2$ such that $0 \in \Omega$ and $\mathbb{R}^2 \backslash \bar{\Omega}$ is connected. There exist harmonic polynomials $H_n \in \mathcal{H}_n$ with the following properties*:

(i) $\sum_{n=0}^{\infty} H_n \in \mathcal{PE}(\bar{\Omega})$;

(ii) *for every compact set K such that $\hat{K} \subset \mathbb{R}^2 \backslash \bar{\Omega}$ and $\partial \hat{K} = \partial K$ and every function $h \in C(K) \cap \mathscr{H}(K^\circ)$, there exists an increasing sequence (n_k) such that* (1) *holds uniformly on K.*

Note that the hypotheses on K are obviously satisfied if $K \subset \mathbb{R}^2 \backslash \bar{\Omega}$ and $\mathbb{R}^2 \backslash K$ is connected. When $N = 2$ conditions (a), (b), (a′), (b′) of Lemmas 2,3 and the condition $\partial \hat{K} = \partial K$ are all mutually equivalent (see [7, Corollary 1.16]), so Corollary 2 is a reformulation of Corollary 1 in the case where $N = 2$.

Corollary 2 does not extend to higher dimensions. To prove this, note first that there is a compact set L in $\mathbb{R}^2$ such that $L^\circ = \emptyset$ and yet $\mathbb{R}^2 \backslash L$ is thin at some point of L (see [6, Example 1.2]). Define $K = L \times [0,1]^{N-2}$, where $N \geqslant 3$. Then K is a compact subset of $\mathbb{R}^N$, $K^\circ = \emptyset, \mathbb{R}^N \backslash K$ is connected, and $\mathbb{R}^N \backslash K$ is thin at some point of K. A proof of this last assertion can be given by using [2, Theorem 7.8.6] and induction on N. By translating, if necessary, we may suppose that $0 \notin K$. If Corollary 2 were true in $\mathbb{R}^N$, then every element of $C(K) \cap \mathscr{H}(K^\circ)$ would be uniformly approximable on K by harmonic polynomials. But such approximation is impossible, since K does not satisfy condition (b′) of Lemma 3.

We conclude with a variant of Theorem 1 in which the set Ω does not appear; it corresponds to the case $r = 0$ of Theorem A.

THEOREM 1′. *There exist harmonic polynomials $H_n \in \mathscr{H}_n$ such that for every compact subset K of $\mathbb{R}^N \backslash \{0\}$ with connected complement and every $h \in \mathscr{H}(K)$ there exists an increasing sequence (n_k) such that* (1) *holds uniformly on K.*

We omit explicit statements of corollaries of Theorem 1′ corresponding to Corollaries 1 and 2 above.

We indicate the changes in the proof of Theorem 1 that are required in order to prove Theorem 1′. The following result is a consequence of Lemma 1.

LEMMA 1′. *Let K be a compact subset of $\mathbb{R}^N \backslash \{0\}$ such that $\mathbb{R}^N \backslash K$ is connected. If $\varepsilon > 0$, $R > 0$, $\mu \in \mathbb{N}$ and $G \in \mathscr{H}(\mathbb{R}^N)$, then there exists a harmonic polynomial $F = \sum_{j=0}^{m} f_j$, where $f_j \in \mathscr{H}_j$ and $m > \mu$, such that* (2) *and* (4) *hold.*

Now let the sequence (Q_n, L_n) of pairs be as in the proof of Theorem 1, except that in defining the class $\mathscr{K}$, to which the compact sets L_n belong, we replace the requirement that $K \subset \mathbb{R}^N \backslash \bar{\Omega}$ by $K \subset \mathbb{R}^N \backslash \{0\}$. Using Lemma 1′, we can show by a recursive construction similar to that in the proof of Theorem 1 that for each $\lambda \in \mathbb{N}$ there exist $m_\lambda \in \mathbb{N}$ and $F_\lambda = f_{\lambda,0} + \cdots + f_{\lambda,m_\lambda}$, where $f_{\lambda,n} \in \mathscr{H}_n$, such that (m_λ) is increasing and (9) and (11) hold when $\lambda \geqslant 1$. Defining $f_{\lambda,n} = 0$ when $n > m_\lambda$ and then defining H_n as in (13), we again find that $H_n \in \mathscr{H}_n$. The proof of Theorem 1′ can now be completed by arguments similar to those in the closing paragraphs of Section 2.

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